April 13, 2017

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To prove two spaces are homeomorphic

$$X \xrightarrow{\boxed{\text{homeomorphism}}} Y$$

we need to find it.

To prove two spaces are not homeomorphic, we usually use contradiction or contrapositive.

Examples

① $(\mathbb{R}, \text{std}) \neq (S^1, \text{std})$

noncompact compact

② $([0,1], \text{std}) \neq (S^1, \text{std})$

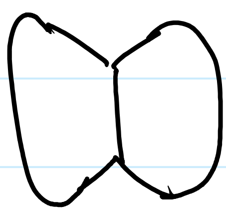
both compact

$\exists\, x \in [0,1]$, $[0,1] \setminus \{x\}$ is disconnected

$\forall\, y \in S^1$, $S^1 \setminus \{y\}$ is connected.

Exercise.

* Justify the above, i.e., if X, Y are homeomorphic and $\exists\, x \in X$ such that $X \setminus \{x\}$ is disconnected, then $\exists\, y \in Y$ such that $Y \setminus \{y\}$ is disconnected.

*  $\neq S^1$

General Principle

Find a topological property P, i.e.,

$$\boxed{X \text{ satisfies } P \Rightarrow h(X) \text{ also satisfies } P \quad \forall \text{ homeomorphism } h}$$

Above examples

① P: compactness

② P: connectedness

③ P: $\exists\, x_0 \in X$, $X \setminus \{x_0\}$ is disconnected

Written Mathematically, they become

① $k : \{\text{Topological spaces}\} \longrightarrow \{\pm 1\}$

$$k(X) = \begin{cases} 1 & X \text{ is compact} \\ -1 & X \text{ is non-compact} \end{cases}$$

$$k(\mathbb{R}) \neq k(S^1) \Rightarrow \mathbb{R} \neq S^1$$

② $c : \{\text{Topological spaces}\} \longrightarrow \mathbb{N} \cup \{\aleph\}$

$$c(X) = \#\text{ of connected components}$$

$$c([0,1]) = 1 = c(S^1) \quad \text{no conclusion}$$

③ $s : \{\text{Topological spaces}\} \longrightarrow \mathbb{N} \cup \{\aleph\}$

$$s(X) = \sup\{c(X \setminus \{x\}) : x \in X\}$$

$$s([0,1]) = 2 \neq 1 = s(S^1) \Rightarrow [0,1] \neq S^1$$

A topological invariant is a function

$$\iota : \left\{ \begin{matrix} \text{Topological} \\ \text{spaces} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{numbers,} \\ \text{matrices,} \\ \text{polynomials,} \\ \text{vector spaces, etc} \end{matrix} \right\}$$

such that

$$X \cong Y \implies \iota(X) = \iota(Y)$$

Euler Characteristic

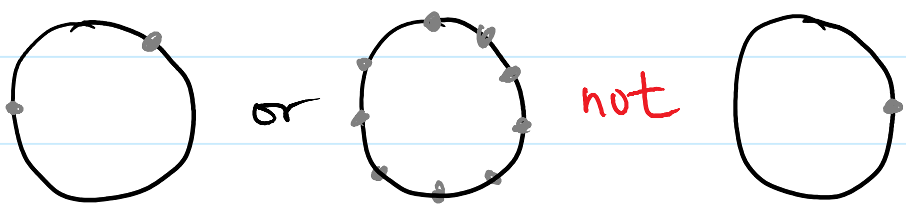
$$\chi : \left\{ \begin{matrix} \text{Topological} \\ \text{spaces} \end{matrix} \right\} \longrightarrow \mathbb{Z}$$

$$\chi(X) \overset{\text{Roughly}}{=\!=} \begin{cases} V - E & \text{if } X \text{ is 1-dim} \\ V - E + F & \text{if } X \text{ is 2-dim} \\ V - E + F - \underset{\text{solid}}{S} & \text{if } X \text{ is 3-dim} \\ \sum_{k=0}^{n} (-1)^k b_k & \text{if } X \text{ is n-dim} \end{cases}$$

- $X = [0,1]$

$$2 - 1 = \chi([0,1]) = 6 - 5$$

- $X = S^1$



$$2 - 2 = \chi(S^1) = 9 - 9$$

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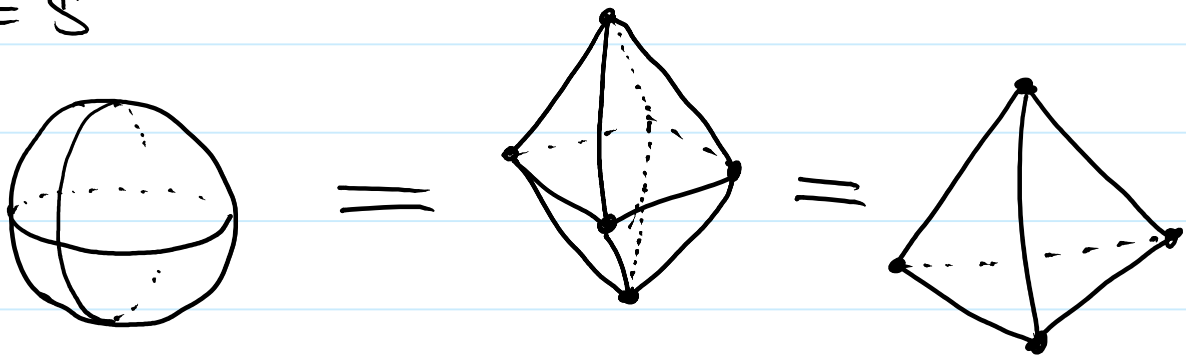
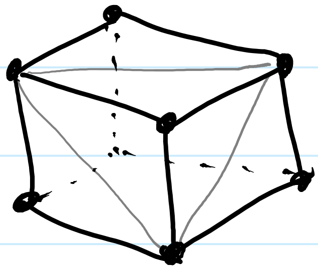
- $X = S^2$



$$6-12+8 = \chi(S^2) = 4-6+4$$



$$8-12+6$$

$20-30+12$ Dodecahedron

$12-30+20$ Icosahedron

- $X = B^3$, the solid ball

$$\chi(B^3) = \chi(S^2) - \text{solid} = 2-1 = 1$$

- The above examples are all compact, so V, E, F, etc are finite numbers.

  Explore inductively and get a value for

$$\chi(\mathbb{R}) = \chi(\mathbb{R}^2) = \chi(\mathbb{R}^n)$$

non-compact

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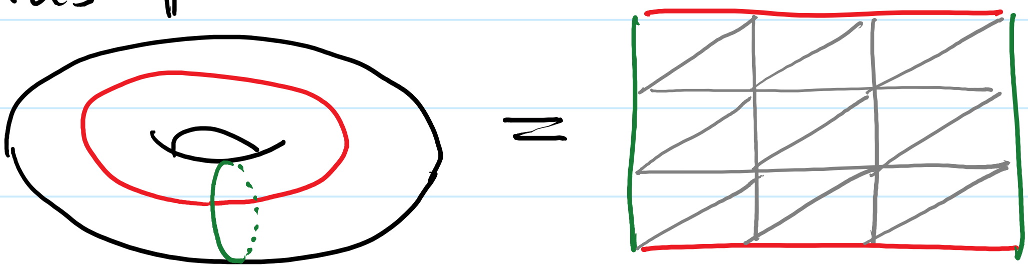
- Torus $\mathbb{T}^2$



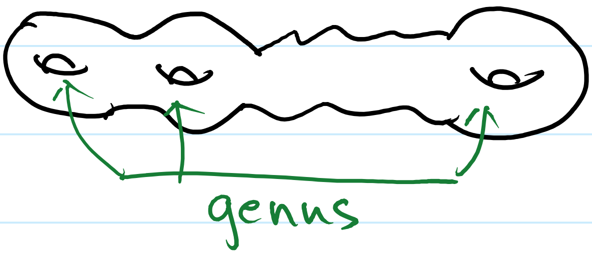
$$\chi(\mathbb{T}^2) = 9 - 27 + 18 = 0$$

Remark: $\mathbb{T}^2 = S^1 \times S^1$

Fact: $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$

Advantage of topological invariants: there is algebraic relation; can be calculated.

- Compact orientable surfaces



$\chi = 2 - 2g$

- Other non-orientable surfaces

$\chi(\mathbb{RP}^2) = 1, \quad \chi(\text{Klein}) = -1$

Fact. Let X, Y be compact surfaces

$$X \cong Y \iff \chi(X) = \chi(Y)$$

No such nice fact in higher dimensions.